

# TOTAL VERTEX IRREGULARITY STRENGTH OF FORESTS

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**ABSTRACT.** We investigate a graph parameter called the *total vertex irregularity strength* ( $tvs(G)$ ), i.e. the minimal  $s$  such that there is a labeling  $w : E(G) \cup V(G) \rightarrow \{1, 2, \dots, s\}$  of the edges and vertices of  $G$  giving distinct weighted degrees  $wt_G(v) := w(v) + \sum_{e \in E(G)} w(e)$  for every pair of vertices of  $G$ . We prove that  $tvs(F) = \lceil (n_1 + 1)/2 \rceil$  for every forest  $F$  with no vertices of degree 2 and no isolated vertices, where  $n_1$  is the number of pendant vertices in  $F$ . Stronger results for trees were recently proved by Nurdin et al.

## 1. INTRODUCTION

Let us consider the simple undirected graph  $G = (V(G), E(G))$  without loops, without isolated edges and with at most one isolated vertex. We assign a label (a natural positive number) to every edge (denoted by  $w(e)$  for all  $e \in E(G)$ ) and to every vertex (denoted by  $w(v)$  for all  $v \in V(G)$ ). We will refer to such a labeling as a *total weighting* of  $G$ . For every vertex  $v \in V(G)$ , we define its *weighted degree* as

$$wt_G(v) = \sum_{e \ni v} w(e) + w(v).$$

We call the labeling  $w$  *irregular* if for each pair of vertices, their weighted degrees are distinct. In [3], a new graph parameter called *total vertex irregularity strength* ( $tvs(G)$ ) was defined as the smallest integer  $s$  such that there exists a total weighting of  $G$  with integers  $\{1, 2, \dots, s\}$  that is irregular. This parameter is similar to the irregularity strength of  $G$  ( $s(G)$ ), introduced in [4] (see also [2], [5], [6] and [7]), where only weights on the edges are allowed.

In [3], several bounds and exact values of  $tvs(G)$  were established for different types of graphs. In particular, the authors proved that for every graph  $G$  with  $n$  vertices and  $m$  edges, the following bounds hold.

$$\left\lceil \frac{n + \delta(G)}{\Delta(G) + 1} \right\rceil \leq tvs(G) \leq n + \Delta(G) - 2\delta(G) + 1, \quad (1)$$

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where  $\Delta(G)$  and  $\delta(G)$  are the maximum and the minimum degree of  $G$ , respectively.

Recently, a much stronger upper bound on  $tv_s(G)$  has been established in [1]. Namely, for every graph  $G$  with  $\delta(G) > 0$ ,

$$tv_s(G) \leq \left\lceil \frac{3n}{\delta} \right\rceil + 1.$$

One should also mention that in [3], exact values of  $tv_s(G)$  for stars, cliques and prisms are given. Furthermore it is shown that for every tree  $T$  without vertices of degree two, the following bounds hold.

$$\left\lceil \frac{n_1+1}{2} \right\rceil \leq tv_s(T) \leq n_1, \quad (2)$$

where  $n_1$  is the number of pendant vertices of  $T$ .

Our main result stated below shows that the trivial lower bound in (2) is the true value of  $tv_s(T)$ . Note the easy observation that for trees with less than 4 vertices, the equality  $tv_s(T) = \left\lceil \frac{n_1+1}{2} \right\rceil$  holds.

**Theorem 1.** *For every forest  $F$  with  $n_1$  vertices of degree one and with no vertices of degree two and no isolated vertices,*

$$tv_s(F) = \left\lceil \frac{n_1+1}{2} \right\rceil.$$

The proof of Theorem 1 is given in the next section.

**Remark.** *Recently, stronger results for trees were proved by Nurdin et al. ([8]). However we decided to publish our paper for two reasons. Firstly, we consider more general case of forests, not only trees. Secondly, we use different proof technique.*

## 2. PROOF OF THEOREM 1

Let us consider a forest  $F$ . Denote by  $V_i$  the set of vertices of degree  $i$ , and let  $n_i = |V_i|$ . So  $V_1$  is the set of pendant vertices and we call edges incident to pendant vertices *pendant edges*. Denote by  $C_{ij}$  the set of its vertices of degree  $i$  with exactly  $j$  pendant neighbors, where  $i, j \geq 0$ , and let  $n_{ij} = |C_{ij}|$ .

Assume that in a total weighting of  $F$  we use labels (weights) from the set  $\{1, 2, \dots, s\}$ . Then, the lowest and the highest weighted degree of a pendant vertex  $v \in V_1$  can be 2 and  $2s$ , respectively. Since there are  $n_1$  such vertices, and each vertex has to have different total weight, the lower bound of (2) trivially follows and extends to all forests, i.e.,

$$tv_s(F) \geq \left\lceil \frac{n_1+1}{2} \right\rceil. \quad (3)$$

To prove Theorem 1, it is sufficient to construct an irregular labeling of  $F$  using elements from the set  $\{1, 2, \dots, s\}$  only, where  $s := \left\lceil \frac{n_1+1}{2} \right\rceil$ .

We will often use the well-known fact that in every tree  $T$  with maximum degree  $\Delta \geq 1$  the numbers  $n_j(T)$  of vertices of degree  $j$  satisfy the equation

$$n_1(T) = 2 + \sum_{j=3}^{\Delta} (j-2)n_j(T). \quad (4)$$

Further, for forests  $F$  with  $n \geq 3$  and  $n_2 \leq 1$ , we have

$$2n_{30} + n_{43} + 2n_{44} \leq n_1 - 2,$$

with equality only for  $K_{1,4}$  and  $P_3$ . This can be seen as follows. For the sake of contradiction, assume that  $F$  is a minimal counter example to the inequality. Then  $n_{43} = 0$  as otherwise we could delete a pendant vertex adjacent to a vertex of class  $C_{43}$  and receive a smaller counter example. Further,  $n_{44} = 0$  as otherwise we can delete a vertex of  $C_{44}$  and its neighbors and either receive a smaller counterexample, or  $F$  itself is a  $K_{1,4}$ , which is not a counterexample. Now delete all pendant vertices from  $F$  to construct a forest  $F'$  with  $n'_3 \geq n_{30}$  and  $n' = n - n_1$  vertices. Then

$$2n_{30} \leq 2n'_3 \leq_{(4)} n' - 2 = n - n_1 - 2 \leq_{(4)} n_1 + n_2 - 4 \leq n_1 - 3.$$

As Theorem 1 is easily verified for  $K_{1,4}$  and  $P_3$ , we may later work with the inequality

$$2n_{30} + n_{43} + 2n_{44} \leq n_1 - 3. \quad (5)$$

Label all non-pendant edges with  $s$ . Next, we will label the pendant edges. First, label half the isolated edges (rounded down) with 1 and the remaining isolated edges with  $s$ . Let now  $v \in C_{kj}$  for some  $k \geq 3$  and  $j \geq 1$ , and we will label the incident pendant edges. Order the values  $\{1, \dots, s-1\}$  as a list  $S = (1, s-1, 2, s-2, 3, s-3, \dots)$ .

- (i) If  $j$  is even, label  $j/2$  pendant edges with  $s$ .
- (ii) If  $j$  is odd, label  $(j-1)/2$  (variant 1) or  $(j+1)/2$  (variant 2) pendant edges with  $s$ .
- (iii) The remaining pendant  $2i + \delta$  ( $0 \leq \delta \leq 1$ ) edges are labeled from  $S$ , where we use the first  $2i$  and the last  $\delta$  values in  $S$ , which have not previously been used on non-isolated edges.

During the process, choose variant 1 and variant 2, so that the number of pendant edges labeled  $s$  is maximized but at most  $s$ , so there are  $s-2$  or  $s-1$  pendant edges labeled with a number less than  $s$ .

For notation, we write  $C_{kj}^i$  for the vertices of variant  $i$  in  $C_{kj}$ , and  $n_{kj}^i$  for their number.

Note that in this labeling, regardless of the labels in  $\{1, \dots, s\}$  we give to the vertices themselves, vertices in  $C_1$  have total weights between 2 and  $2s$ , and all other vertices have total weights of at least  $2s + 2$ . Label every pendant vertex incident to a non-isolated edge labeled with a number less than  $s$  with 1. This guarantees that the total weights of all these vertices are different. The remaining vertices in  $C_1$  can now be weighted greedily one-by-one.

For the weight range  $2s + 2 \leq wt(v) \leq 3s$ , only vertices in  $C_{31}^1 \cup C_{32} \cup C_{33}$  play a role. Vertices  $v \in C_{33}^2$  have total weight  $2s + w(v)$ , all others have weight  $2s + w(e) + w(v)$ , where  $e$  is a pendant edge with label other than  $s$  incident to  $v$ . We can set  $w(v) = 1$  for at most  $s - 1 - n_{33}^2$  vertices  $v \in (C_{31}^1 \cup C_{32} \cup C_{33}^1)$ , giving them all different total weights in this weight range, and greedily choose weights for the vertices in  $C_{33}^2$  to fill out the remaining weights. Note that  $n_{33}^2 \leq \frac{n_{33}+1}{2} \leq \frac{n_1+3}{6} \leq \frac{s+1}{3}$ , so this is possible.

For the weight range  $3s + 1 \leq wt(v) \leq 4s$ , only vertices in

$$C_3 \cup C_{41}^1 \cup C_{42} \cup C_{43} \cup C_{44} \cup C_{55}^1$$

play a role. Label all vertices  $v \in C_{41}^1 \cup C_{42} \cup C_{43}^2 \cup C_{55}^1$  with  $w(v) = s$ , giving them total weight  $wt(v) = 4s + w(e)$ , and thus pairwise different weights outside the weight range we currently consider. We give the remaining (at most)  $\max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\}$  vertices from  $C_{31}^1 \cup C_{32} \cup C_{33}^1$  weight  $w(v) = s$ , giving them all different total weights  $3s + w(e)$  in this range. Vertices  $v \in C_{30} \cup C_{43}^1 \cup C_{44}$  have total weight  $3s + w(v)$ , so one can greedily fit them into the remaining weights of the range, provided there is enough room. If  $n_{31} + n_{32} + n_{33} \geq s$ , then

$$\begin{aligned} n_{30} + n_{43}^1 + n_{44} + \max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\} \\ \leq n_3 + n_4 - s + 1 \stackrel{(4)}{\leq} n_1 - s - 1 \leq s - 1. \end{aligned}$$

If, on the other hand,  $n_{31} + n_{32} + n_{33} \leq s - 1$ , then

$$\begin{aligned} n_{30} + n_{43}^1 + n_{44} + \max\{1, n_{31}^1 + n_{32} + n_{33} - s + 1\} \\ = n_{30} + n_{43}^1 + n_{44} + 1 \stackrel{(5)}{\leq} \left\lfloor \frac{n_1}{2} \right\rfloor = s - 1. \end{aligned}$$

Thus, there is enough room to fit all the mentioned vertices.

For the weight range  $wt(v) \geq 4s + 1$ , only vertices in  $C_j, j \geq 4$  play a role. By (4), there are at most  $s$  such vertices. We have dealt with vertices in  $C_4 \setminus C_{40}$  already, so all remaining vertices have  $wt(v) \geq 4s + w(v)$ . Thus, we have at each vertex  $s$  choices for the total weight, which is enough to allow us to greedily pick the values  $w(v)$  to complete the irregular total weighting.  $\square$

## 3. FINAL REMARKS

Note that in the proof of Theorem 1, the total weight  $2s + 1$  was not used. With this observation we can prove a slight generalization.

**Theorem 1.** *For every forest  $F$  on  $n$  vertices, with  $n_0 = 0$  and  $n_2 \leq 1$ , we have*

$$tvs(F) = \left\lceil \frac{n_1+1}{2} \right\rceil.$$

*Proof.* The proof is the same as above, with one extra observations. Just note in the end, that since a vertex  $v$  of degree 2 is incident to an edge of label  $s$  (either a non-pendant edge or a pendant edge with this label by the construction), we can choose  $w(v)$  such that  $wt(v) = 2s + 1$ .  $\square$

As a consequence we have the following corollary.

**Corollary 3.1.** *Let  $T$  be a binary tree. Then  $tvs(T) = \left\lceil \frac{n_1+1}{2} \right\rceil$ .*

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